# Bayesian estimation for collisional thermometry

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## How to measure temperature?

☞ Important for ultra-low temperature quantum experiments [1]

[1]V. Mukherjee, A. Zwick, A. Ghosh, X. Chen, and G. Kurizki, Communications Physics 2, 162 (2019), arXiv:1711.09660.

## How to measure temperature?

- Important for ultra-low temperature quantum experiments [1]
- Temperature is not an observable
- It must be *indirectly inferred*
- It is encoded into states and/or operators

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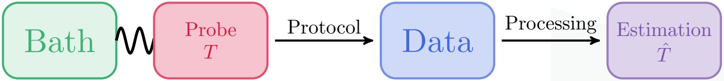


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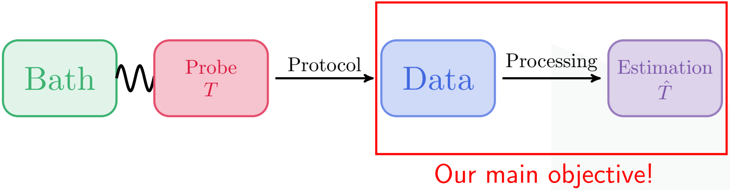


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## The Model

Open quantum system approach [2]

☞ Master equation → system-environment interaction

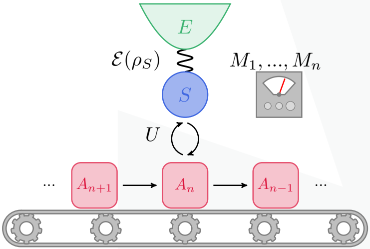
$$\frac{d\rho_S}{dt} = \mathcal{E}(\rho_S) = \gamma(\bar{n}+1)\mathcal{D}[\sigma_-^S] + \gamma\bar{n}\mathcal{D}[\sigma_+^S]$$

with $\bar{n} = (e^{\Omega/T} - 1)^{-1}$ and

$$\mathcal{D}[L] = L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}$$

Map:

$$\mathcal{E}(\rho_S) = e^{\tau_{SE}\mathcal{L}}(\rho_S) \qquad (1)$$

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## The Model

Open quantum system approach [2]

☞ Partial-SWAP → system-ancilla interaction

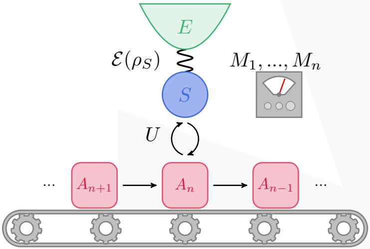
$$\mathcal{U}_{SA_n} = \exp\left\{-i\tau_{SA}g(\sigma_+^S\sigma_-^{A_n}+\sigma_-^S\sigma_+^{A_n})\right\}$$

Ancillas and System → Qubits

$$H = \frac{\Omega}{2}\sigma_Z$$

Ground-state ancillas

$$\rho_A = |0\rangle\langle 0|$$

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## The Model

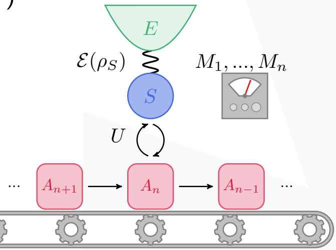
Open quantum system approach [2]

☞ Combined SE + SA interaction → Stroboscopic Map

$$\rho_S^n = \mathrm{tr}_{A_n}\{\mathcal{U}_{SA_n}\mathcal{E}(\rho_S^{n-1}\otimes\rho_A^0)\mathcal{U}_{SA_n}^\dagger\} := \Phi(\rho_S^{n-1})$$

Repeated interactions. The system reaches the SS:

$$\rho_S^* = \Phi(\rho_S^*)$$

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## The Model

Open quantum system approach [2]

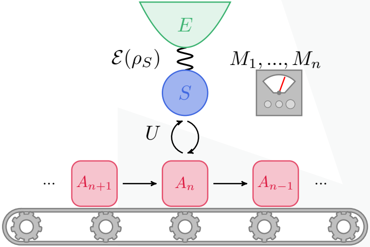
☞ Single ancilla measurements → POVMs $\{M_i\}$

$$p(X_n, ..., X_1|T) \approx p(X_n|T)...p(X_1|T)$$

i.i.d. measurements approximation, where

$$p(X_i|T) = \mathrm{tr}(M_{X_i}\rho_{A_i}) \qquad (1)$$

and $X_i = 0, 1$ (Comp. Basis)

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## How to perform parameter inference?

Probability distribution for the temperature ($T$ is considered a random variable)

**Bayes Theorem**

We construct a posterior distribution $P(T|\boldsymbol{X})$ from a prior $P(T)$

$$P(T|\boldsymbol{X}) = \frac{P(\boldsymbol{X}|T)P(T)}{P(\boldsymbol{X})}, \tag{2}$$

given the outcomes $\boldsymbol{X} = (X_1, ..., X_n)$.[3]

[3] E. L. Lehmann and G. Casella, Theory of Point Estimation, 2nd ed., Springer Texts in Statistics (Springer-Verlag, New York, 1998)

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Likelihood, generates $(X_1, ..., X_n)$

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## First results

Collisional model outcomes (for $g\tau_{SA} = \pi/2$):

$$p_1 := p(X_i = 1|T) = \frac{1 - e^{-\Gamma}}{1 + e^{\frac{\Omega}{T}}} \tag{3}$$

Thermal relaxation parameter $\Gamma = \gamma(2n+1)\tau_{SE}$ and $p_0 = 1 - p_1$. We begin with a "flat prior" distribution. The true value is $T_0 = 1.5$.

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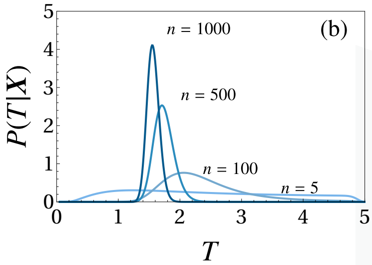

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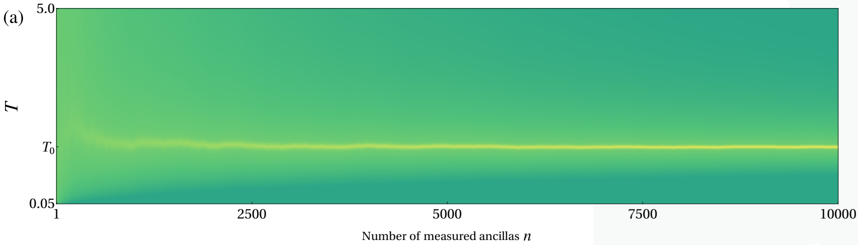

# Bayesian estimators

We also want a number, not a distribution!

**Bayesian Average**

The posterior mean

$$\hat{T}(\boldsymbol{X}) = \int T P(T|\boldsymbol{X}) dT \tag{4}$$

is the estimator which minimizes the Bayesian mean-squared error:

$$\epsilon_B(\hat{T}(\boldsymbol{X})) = \int P(T) dT \int (T - \hat{T})^2 P(\boldsymbol{X}|T) d\boldsymbol{X} \tag{5}$$

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Compare it with the usual MSE: $\epsilon(\hat{T}(\boldsymbol{X})|T) = \int (T - \hat{T})^2 P(\boldsymbol{X}|T) d\boldsymbol{X}$

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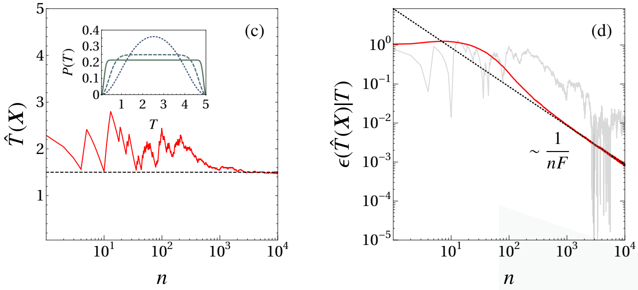
Integral over $T$!

The integrated error is temperature-independent! [4]

[4] Harry L. Van. Trees and Kristine L. Bell. Detection Estimation and Modulation Theory, 2nd Edition, Part I. John Wiley & Sons, 2013

## Estimation and the MSE

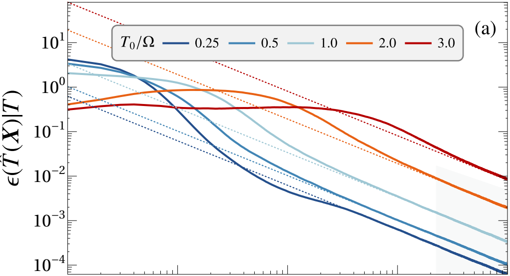
Posterior mean (left) and the mean-squared error (right) for $T_0/\Omega = 1.5$:




The MSE converges to the Cramér-Rao bound $1/nF(T)$! The Fisher Information is given by: $F(T) = \sum_i \frac{1}{p_i}(\partial p_i/\partial T)^2$.

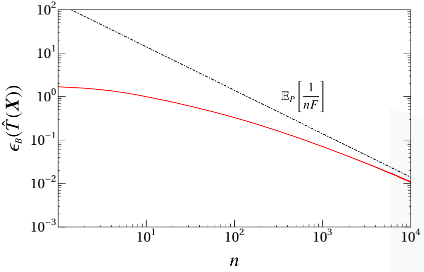
# Results

- The performance depends on the temperature



# Results

- The performance depends on the temperature
- We can integrate the MSE over $\int \ldots P(T)dT$ to get the Bayesian MSE
- The analysis here is now "temperature-independent"



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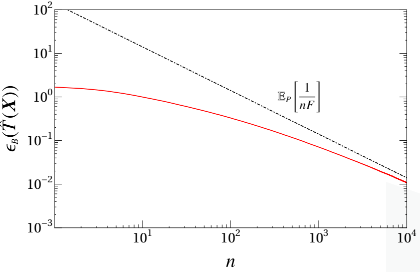


The Bayesian MSE does not converges to $1/nF(T)$, which is temperature dependent, but rather to its average $\int 1/nF(T) \times P(T)dT$ over the prior!

# Conclusion

- ☞ Collisional models provide a scalable and suitable platform to perform thermometry
- ☞ Bayesian updating can be used to construct posterior distributions for the temperature
- ☞ Bayesian inference can be used to pragmatically construct estimators
- ☞ These estimators saturate the CRB asymptotically
- ☞ We can both use figures of merit and develop optimal strategies which are temperature-independent

# Thank you!

## References


- Gabriel O. Alves, Gabriel T. Landi, arXiv:2106.12072
- Edwin T. Jaynes and G. Larry Bretthorst. Probability theory: the logic of science. Cambridge Univ. Press, 2019
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- Y. Li, L. Pezz, M. Gessner, Z. Ren, W. Li, and A. Smerzi, Entropy 20, 25 (2018), arXiv:1804.10048.

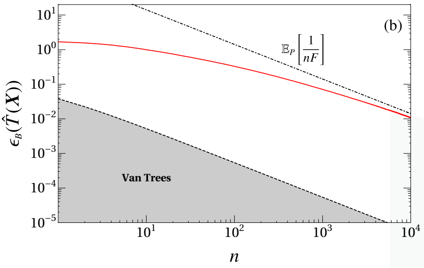

FAPESP CAPES

# Appendix - Van Trees-Schützenberger inequality

☞ A bayesian analogue of the Cramér-Rao Bound[5]

$$\epsilon_B(\hat{T}(\boldsymbol{X})) \geq \frac{1}{\mathbb{E}_P[F(T)] + F_P}, \tag{6}$$

where $F_P = \int P(T)\left(\frac{\partial \ln P(T)}{\partial T}\right)^2 dT$ and $\mathbb{E}_P[...] = \int P(T)...dT$

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# Bayesian MSE - changing the parameters

- An analysis which is independent of the temperature
- Prior and interval size change the optimal parameters in the Bayesian sense
- A temperature increase shifts the optimal SE coupling to the left